Malliavin calculus and ergodic properties of highly degenerate 2D stochastic Navier–Stokes equation

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Abstract

The objective of this note is to present the results from the two papers [25] and [16]. We study the Navier–Stokes equation on the two–dimensional torus when forced by a finite dimensional white Gaussian noise. We give conditions under which both the law of the solution at any time $t > 0$, projected on a finite dimensional subspace, has a smooth density with respect to Lebesgue measure and the solution itself is ergodic. In particular, our results hold for specific choices of four dimensional white Gaussian noise. Under additional assumptions, we show that the preceding density is everywhere strictly positive.

Résumé

Le but de cette Note est d'annoncer les résultats contenus dans les articles [25] et [16]. Nous étudions l'équation de Navier–Stokes sur le tore bidimensionnel forcée par un bruit blanc gaussien de dimension finie. Nous donnons des conditions sous lesquelles d'une part la loi de la solution à tout instant $t > 0$, projetée sur un espace de dimension finie, a une densité régulière par rapport à la mesure de Lebesgue (qui, sous des hypothèses supplémentaires, est strictement positive partout), et d'autre part la solution de la même équation est un processus ergodique. En particulier ces résultats sont vrais dans certains cas de bruit blanc gaussien de dimension quatre.

1. Introduction

This note reports on recent progress made in [25,16] on the study of the two dimensional Navier–Stokes equation driven by an additive stochastic forcing. Recall that the Navier–Stokes equation describes the time evolution of an incompressible fluid. In vorticity form, it is given by

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$$
\begin{cases} \dfrac{\partial w}{\partial t}(t,x) + B(w,w)(t,x) = \nu \Delta w(t,x) + \dfrac{\partial W}{\partial t}(t,x) \\ w(0,x) = w_0(x), \end{cases} \tag{1}
$$

where $x = (x_1, x_2) \in \mathbb{T}^2$, the two-dimensional torus $[0, 2\pi] \times [0, 2\pi]$, $\nu > 0$ is the viscosity constant, $\frac{\partial W}{\partial t}$ is a white-in-time stochastic forcing to be specified below, and

$$
B(w, \tilde{w})(x) = \sum_{i=1}^{2} (\mathcal{K}w)_i(x) \frac{\partial \tilde{w}}{\partial x_i}(x) \ ,
$$

where $\mathcal{K}$ is the Biot-Savart integral operator which will be defined next. First, we define a convenient basis in which we will perform all explicit calculations. Setting $\mathbb{Z}^2_+ = \{(j_1, j_2) \in \mathbb{Z}^2 : j_2 > 0\} \cup \{(j_1, j_2) \in \mathbb{Z}^2 : j_1 > 0, j_2 = 0\}$, $\mathbb{Z}^2_- = -\mathbb{Z}^2_+$ and $\mathbb{Z}^2_0 = \mathbb{Z}^2_+ \cup \mathbb{Z}^2_-$, we define a real Fourier basis for functions on $\mathbb{T}^2$ with zero spatial mean by

$$
e_k(x) = \begin{cases} \sin(k \cdot x) & k \in \mathbb{Z}^2_+ \\ \cos(k \cdot x) & k \in \mathbb{Z}^2_- \ . \end{cases}
$$

Write $w(t, x) = \sum_{k \in \mathbb{Z}^2_0} \alpha_k(t) e_k(x)$ for the expansion of the solution in this basis. With this notation, in the two-dimensional periodic setting,

$$
\mathcal{K}(w) = \sum_{k \in \mathbb{Z}^2_0} \frac{k^\perp}{|k|^2} \alpha_k e_{-k}, \tag{2}
$$

where $k^\perp = (-k_2, k_1)$. See for example [20] for more details on the deterministic vorticity formulation in a periodic domain. We use the vorticity formulation for simplicity, but all of our results can easily be translated into statements about the velocity formulation of the problem. We solve (1) on the space $\mathbb{L}^2 = \{f = \sum_{k \in \mathbb{Z}^2_0} a_k e_k : \sum |a_k|^2 < \infty\}$. For $f = \sum_{k \in \mathbb{Z}^2_0} a_k e_k$, we define the norms $\|f\|^2 = \sum |a_k|^2$ and $\|f\|_1^2 = \sum |k|^2 |a_k|^2$.

The emphasis of this note will be on forcing which directly excites only a few degrees of freedom. Such forcing is both of primary modeling interest and is technically the most difficult. Specifically we consider forcing of the form

$$
W(t, x) = \sum_{k \in \mathcal{Z}_*} \sigma_k W_k(t) e_k(x) \ . \tag{3}
$$

Here $\mathcal{Z}_*$ is a finite subset of $\mathbb{Z}^2_0$, $\sigma_k > 0$, and $\{W_k : k \in \mathcal{Z}_*\}$ is a collection of mutually independent standard scalar Brownian Motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The note describes two sets of results contained in two papers by two different subsets of the authors. In the first, Mattingly and Pardoux [25] give conditions ensuring that any projection of the time t transition probability of the solution of (1) onto a finite dimensional subspace has a C^∞ density with respect to Lebesgue measure. The result is based on the Malliavin calculus. Under additional conditions, this density is shown to be everywhere positive. The techniques developed are quite general and we expect they can be applied to many nonlinear, stochastic partial differential equations with additive noise. These results provide a first step towards a truly infinite dimensional version of Hörmanders celebrated "sum of squares" theorem [17].

In the second paper, Hairer and Mattingly [16] give necessary and sufficient conditions for the main results and estimates of [25] to hold. They then use these tools to build a theory which, when applied to (1), proves that it has a unique invariant measure under extremely general and essentially sharp assumptions. In addition to the tools from [25], they introduce new concept and tool which together provide an abstract framework in which the ergodicity of (1) is proven. The concept is a generalization

of the strong Feller property for a Markov process which, for reasons that will be made clear below, is called the *asymptotic strong Feller* property. The main feature of this property is that a diffusion which is irreducible and asymptotically strong Feller can have at most one invariant measure. It thus yields a natural generalization of Doob's theorem. The tool is an approximate integration by parts formula, in the sense of Malliavin calculus, which is used to prove that the system enjoys the asymptotic strong Feller property. To the best of the authors knowledge, this paper is the first to prove ergodicity of a nonlinear stochastic partial differential equation (SPDE) under assumptions comparable to those assumed when studying finite dimensional stochastic differential equations.

The ergodic theory of infinite dimensional stochastic systems, and SPDEs specifically, has been a topic of intense study over the last two decades. Until recently, the forcing was always assumed to be elliptic and spatially rough. In our context this translates to $\mathcal{Z}_* = \mathbb{Z}_0^2$ and $|\sigma_k| \sim |k|^{-\alpha}$ for some positive α. Flandoli and Maslowski [13] first proved ergodic results for (1) under such assumptions. This line of inquiry was extended and simplified in [11,14]. They represent a larger body of literature which characterizes the extent to which classical ideas developed for finite dimensional Markov processes apply to infinite dimensional processes. Principally they use tools from infinite dimensional stochastic analysis to prove that the processes are strong Feller in an appropriate topology and then deduce ergodicity.

Next three groups of authors in [18,3,8], contemporaneously greatly expanded the cases known to be ergodic. They use the Foias-Prodi type reduction, first adapted to the stochastic setting in [21] and the pathwise contraction of the high spatial frequencies already used in [22] to prove ergodicity of (1) at sufficiently high viscosity. All of the results hinged on the observation that if all of the unstable directions are stochastically perturbed, then the system could be shown to be ergodic. A general overview of these ideas with simple examples can be found in [24]. These ideas have been continued in a number of papers. See for instance [7,4,23,15,19,24].

Unfortunately, the best current estimates on the number of unstable directions in (1) grow inversely with the viscosity ν. Hence the physically important limit of $\nu \to 0$ while a fixed, finite scale is forced were previously outside the scope of the theory. However there existed strong indications that ergodicity held in this case. Specifically in [9] it was shown that the generator of the diffusion associated to finite dimensional Galerkin approximations of (1) was hypoelliptic in the sense of Hörmander when only a few directions were forced. This hypoellipticity is the crucial ingredient in the proof of ergodicity from [9].

The "correct" ergodic theorem needs to incorporate in its statement information on how the randomness spreads from the few forced directions to all of the unstable directions. This understanding when combined with what had been learned in [21,22,18,3,8] should yield unique ergodicity. This is the program executed in the papers discussed in this note.

2. The Geometry of the Forcing and Cascade of Randomness

The geometry of the forcing is encoded in the structure of $\mathcal{Z}_*$ from (3). As observed in [9], its structure gives information about how the randomness is spread throughout phase space by the nonlinearity.

Define $\mathcal{Z}_0$ to be the symmetric, and hence translationally stationary part of the forcing set $\mathcal{Z}_*$, given by $\mathcal{Z}_0 = \mathcal{Z}_* \cap (-\mathcal{Z}_*)$. Then define the collection

$$\mathcal{Z}_n = \left\{ \ \ell + j \in \mathbb{Z}_0^2 \ : \ j \in \mathcal{Z}_0, \ \ell \in \mathcal{Z}_{n-1} \quad \text{with} \quad \ell^\perp \cdot j \neq 0, \ |j| \neq |\ell| \ \right\}$$

and lastly,

$$\mathcal{Z}_\infty = \bigcup_{n=1}^{\infty} \mathcal{Z}_n .$$

$\mathcal{Z}_\infty$ captures the directions to which the randomness has spread. This can be understood in the following way. Denote by ∂_k the partial derivative into the direction e_k of the phase space and define (on a formal level) the first order differential operator $\mathcal{X}$ by

$$\mathcal{X} = \sum_{k\in\mathbb{Z}_0^2} \big(B(w,w)_k - \nu|k|^2\big)\partial_k \;.$$

Then the generator of the Markov process associated to (1) is formally given by

$$\mathcal{L} = \mathcal{X} + \frac{1}{2}\sum_{k\in\mathcal{Z}_*} \sigma_k\partial_k^2 \;.$$

Note that $B(w,w)_k = \sum_{\ell,j} c_{k,j,\ell} w_\ell w_j$, where $c_{k,j,\ell} \neq 0$ if and only if $k \in \{j \pm \ell, -j \pm \ell\}$ and $\ell^\perp \cdot j \neq 0$, $|j| \neq |\ell|$. Therefore, all differential operators of the type ∂_k with $k \in \mathcal{Z}_\infty$ can be obtained as an iterated Lie bracket of finite length involving X_0 and ∂_ℓ with $\ell \in \mathcal{Z}_*$.

Since we want to ensure that all of the unstable directions are stochastically agitated, we seek conditions where $\mathcal{Z}_\infty = \mathbb{Z}_0^2$. The following essentially sharp characterization of this situation is given in [16].

Proposition 2.1 *One has $\mathcal{Z}_\infty = \mathbb{Z}_0^2$ if and only if both :*

(i) Integer linear combinations of elements of $\mathcal{Z}_0$ generate $\mathbb{Z}_0^2$.

(ii) There exist at least two elements in $\mathcal{Z}_0$ with unequal euclidean norm.

This characterization is sharp in the sense that if $\mathcal{Z}_* = -\mathcal{Z}_*$ and one of the above two conditions fails, then there exists a non-trivial subspace of $\mathbb{L}^2$ which is left invariant under the dynamics of (1). Also notice that if

$$\mathcal{Z}_0 = \{(0,1),(0,-1),(1,1),(-1,-1)\}$$

then Proposition 2.1 implies that $\mathcal{Z}_\infty = \mathbb{Z}_0^2$. Hence forcing four well chosen modes is sufficient to have the randomness move through the entire system. Of course one can also force a small number of modes center elsewhere than at the origin and obtain the same effect. The next two sections discuss the implications of $\mathcal{Z}_\infty = \mathbb{Z}_0^2$.

3. Malliavin Calculus and Densities

We define

$$S_\infty = \mathrm{Span}\big(e_k : k \in \mathcal{Z}_\infty \cup \mathcal{Z}_*\big) \;. \tag{4}$$

One of the main results of [25] is the following :

Theorem 3.1 *For any $t > 0$ and any finite dimensional subspace S of S_∞, the law of the orthogonal projection $\Pi w(t,\cdot)$ of $w(t,\cdot)$ onto S is absolutely continuous with respect to the Lebesgue measure on S and has a C^∞ density.*

In [10], Eckmann and Hairer used Malliavin calculus to prove a version of Hörmander's "sum of squares" theorem for a particular SPDE and deduce ergodicity. However, all of the techniques of that paper required that the forcing excite all but a finite number of directions and that the forcing be spatially rough as in [13,6]. The proof of Theorem 3.1 builds on ideas introduced into Malliavin calculus by Ocone in [26]. The central idea is an alternative representation of the Malliavin matrix of (1) using the time reversed adjoint of the linearization of (1). Ocone used this representation when the SPDE was linear in the initial data and the forcing. When the noise is additive, [25] extends that idea to the nonlinear case.

Let $J_{s,t}\xi$ be the solution of linearization of (1) at time t with initial condition ξ at time s, $s \le t$. Let $\bar{J}^*_{s,t}\xi$ denote the solution to the $\mathbb{L}^2$-adjoint of the linearizion at time s, $s \le t$, with terminal condition ξ

at time t. Since the equation is time reversed, the adjoint is well posed. With this notation, the so–called "Malliavin covariance matrix" $\mathcal{M}_t$ can be represented by

$$\langle \mathcal{M}_t \phi, \phi \rangle = \sum_{k \in \mathcal{Z}_*} \int_0^t \sigma_k^2 \langle J_{s,t} e_k, \phi \rangle^2 \, ds = \sum_{k \in \mathcal{Z}_*} \int_0^t \sigma_k^2 \langle e_k, \bar{J}_{s,t}^* \phi \rangle^2 \, ds$$

where $\phi \in \mathbb{L}^2$. The second of these representations is the one used in [25]. Because of the time reversal, the representation is not adapted to the filtration generated by W and new estimates concerning anticipating stochastic processes are required to obtain the needed estimates. Essentially one needs to show that the Malliavin matrix is non-degenerate on the subspace S and that the moments of the reciprocal of the norm of the Malliavin matrix on this subspace are finite. This is accomplished through the following estimate which also gives information about the separation of the randomness on large and small scales.

Theorem 3.2 *Let Π be the orthogonal projection of $\mathbb{L}^2$ onto a finite dimensional subspace of S_∞. For any $t > 0$, $\eta > 0$, $p \geq 1$, $M > 0$ and $K \in (0,1)$ there exist two constants $c = c(\nu, \eta, p, |\mathcal{Z}_*|, t, K, M, \Pi)$ and $\epsilon_0 = \epsilon_0(\nu, K, |\mathcal{Z}_*|, t, M, \Pi)$ such that for all $\epsilon \in (0, \epsilon_0]$,*

$$\mathbb{P}\Big(\inf_{\phi \in S(M,K,\Pi)} \langle \mathcal{M}_t \phi, \phi \rangle < \epsilon \Big) \leq c \exp(\eta \|w(0)\|^2) \epsilon^p$$

where $S(M, K, \Pi) = \{\phi \in S_\infty : \|\phi\| = 1, \|\phi\|_1 \leq M, \|\Pi\phi\| \geq K\}$.

With additional assumptions on the controllability of (1) conditions are also given ensuring the strict positivity of the density. This extends results of Ben Arous and Léandre [2] and Aida, Kusuoka and Stroock [1] to this setting. We refer the reader to [25] for the exact conditions and the details.

4. Unique Ergodicity

Recall that an *invariant measure* for (1) is a probability measure $\mu_\star$ on $\mathbb{L}^2$ such that $P_t^* \mu_\star = \mu_\star$, where P_t^* is the semigroup on measures dual to the Markov transition semigroup P_t defined by $(P_t \phi)(w) = \mathbb{E}_w \phi(w_t)$ with $\phi \in C_b(\mathbb{L}^2)$. While the existence of an invariant measure for (1) can be proved by "soft" techniques using the regularizing and dissipativity properties of the flow [5,12], showing its uniqueness is a more challenging problem that requires a detailed analysis of the nonlinearity. The importance of showing the uniqueness of $\mu_\star$ is illustrated by the fact that it implies that

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T \phi(w_t) \, dt = \int_{\mathbb{L}^2} \phi(w) \, \mu_\star(dw) \,, \tag{5}$$

for all bounded continuous functions ϕ and all initial conditions $w_0 \in \mathbb{L}^2$. It thus gives some mathematical ground to the *ergodic assumption* usually made in the physics literature when discussing the qualitative behavior of (1). The following theorem is the main result of [16].

Theorem 4.1 *If $\mathcal{Z}_\infty = \mathbb{Z}_0^2$, then,* (1) *has a unique invariant measure in $\mathbb{L}^2$.*

When combined with Proposition 2.1, this theorem gives easy to verify conditions guaranteeing a unique invariant measure.

The concept of a strong Feller Markov process appears to be less useful in infinite dimensions than in finite dimensions. In particular if P_t is strong Feller, then the measures $P_t(u, \cdot)$ and $P_t(v, \cdot)$ are equivalent for all initial conditions $u, v \in \mathbb{L}^2$. It is easy to construct an ergodic SPDE which does not satisfy this property.

Recall the following standard sufficient criteria for P_t to be strong Feller : there exists a locally bounded function $C(w,t)$ such that

$$|\nabla(P_t\phi)(w)| \leq C(w,t)\|\phi\|_\infty$$

for all Fréchet differentiable functions $\phi : \mathbb{L}^2 \to \mathbb{R}$. While we will not give the exact definition of the asymptotic strong Feller property here, the following similar condition implies that the process is asymptotically strong Feller : there exists a locally bounded $C(w)$, a non-decreasing sequence of times t_n, and a strictly decreasing sequence ϵ_n with $\epsilon_n \to 0$ so that

$$|\nabla(P_{t_n}\phi)(w)| \leq C(w)\|\phi\|_\infty + \epsilon_n\|\nabla\phi\|_\infty \tag{6}$$

for all Fréchet differentiable functions $\phi : \mathbb{L}^2 \to \mathbb{R}$ and all $n \geq 1$. In applications one typically has $t_n \to \infty$. Hence, the process behaves as if it acquired the strong Feller property at time infinity, which justifies the term asymptotic strong Feller.

First observe that $\langle \nabla_w(P_t\phi)(w), \xi\rangle = \mathbb{E}_w(\nabla\phi)(w_t)J_{0,t}\xi$. Next we seek a direction v in the Cameron-Martin space so that if $\mathcal{D}^v$ denotes the Malliavin derivative in the direction v then $J_{0,t}\xi = \mathcal{D}^v w_t$. In finite dimensions, we can often do this exactly ; however, in infinite dimensions we only know how to achieve this up to some error. Setting $\rho_t = J_{0,t}\xi - \mathcal{D}^v w_t$, we have the approximate integration by parts formula.

$$\begin{aligned}\mathbb{E}_w(\nabla\phi)(w_t)J_{0,t}\xi &= \mathbb{E}_w\mathcal{D}^v[\phi(w_t)] + \mathbb{E}_w(\nabla\phi)(w_t)\rho_t \\ &= \mathbb{E}_w\phi(w_t)\int_0^t v_s dW_s + \mathbb{E}_w(\nabla\phi)(w_t)\rho_t \ .\end{aligned}$$

From this equality one can quickly deduce (6), provided $\mathbb{E}|\int_0^\infty v_s dW_s| < \infty$ and $\mathbb{E}|\rho_t| \to 0$ as $t \to \infty$. In [16], a v_t is chosen so that these conditions hold. The analysis is complicated by the fact that the v_t constructed there is not adapted to the Brownian filtration. This complication seems unavoidable. Hence, the stochastic integral is a Skorohod integral and all of the calculations are made more complicated.

The ideas developed here can also be used to prove exponential mixing using the ideas from [23,15]. These results will be presented elsewhere.

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